Optimal Laminated Composite Shells for Buckling and Vibration

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Methods are proposed for the determination of the optimal ply angle variation through the thickness of symmetric angle-ply shells of uniform thickness. These methods use continuous piecewise-linear segment approximations or discontinuous piecewise-constant segment approximations to the ply angle function. A mathematical programming (MP) problem is formulated using segment ply angles and thicknesses as design variables. A special MP algorithm, capable of treating multiple objective functions, combined with a critical mode search is used to solve this problem. The procedure is applied to the maximization of the minimum natural frequency or buckling load of a thin, simply supported, circular, cylindrical, angle-ply shell. Results show large performance gains result from the use of optimal variable ply angle configurations, compared to an optimal constant ply angle. The continuously variable ply angle approximation is particularly effective.

Nomenclature

A = extensional stiffness
B = buckling factor
D = bending stiffness
E_1 = Young's modulus of a lamina in the 1 (filament) direction
E_2 = Young's modulus in the 2 (normal) direction
f = objective function
g_j = constraint functions
G_{12} = shear modulus of a lamina in the 1-2 plane
h = shell thickness
I = number of MP problem variables
J = number of constraints
k = number of different approximating segments to the optimal $\phi(z)$ function
k_1,k_2 = axial and lateral shell loading fractions, respectively
$\mathcal{L}$ = problem operator
$\mathcal{L}_k$ = problem operator for problem with k different segments
L = shell length
m = number of axial buckling or vibration half-waves
n = number of circumferential waves
q = buckling load parameter
Q = stiffness matrix at a point
$\bar{Q}$ = overall stiffness matrix
R = shell midplane radius
S = approximating series type
t_i = thickness of the ith segment
X = nondimensional shell length parameter
z = position variable
z_i = position of interface between segment i and segment $i+1$
$\phi(z)$ = ply angle function
ϕ_i = ply angle variable
Φ = optimization problem design variables
ν_{12} = major Poisson's ratio of a lamina
ϵ = convergence specification
$\bar{\omega}$ = nondimensional natural frequency of vibration

Subscripts

i = associated with the ith segment or mode
k = associated with problem $\mathcal{L}_k$

Superscripts

$1,2$ = associated with approximating sequence 1 or 2, respectively
r = associated with approximating sequence 1 or 2
$(\ )^*$ = associated with the optimum

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I. Introduction

A PRINCIPAL advantage of composite materials lies in the ability of the designer to tailor the material properties to the application. For the case of filamentary composites, given a particular composite system, this tailoring normally involves the selection of the appropriate ply angle configuration. In the design of composite structures one is thus lead naturally to the problem of the selection of an appropriate, and preferably, optimal ply angle distribution in the structure. Studies of optimal ply angle variation across the thickness of the angle-ply plates[1-4] demonstrate the substantial nature of the performance improvement possible by use of an optimal variable angle-ply construction. Reference 4 provides the most recent study of optimal ply angle configurations in plates. This study includes minimization of stress under in-plane and out-of-plane loads. For a given thickness and loading condition involving out-of-plane loads a considerable stress reduction from the level found in optimal constant ply angle plates is achieved by the use of optimal variable ply angle configurations. Unfortunately, it is also shown where there are only in-plane loads that neither the stresses in the plates can be reduced nor buckling resistance of plates increased by use of variation in ply angle. This observation is also apparent from the work of Hirano,[2] who studies the buckling resistance of optimal angle-ply multiple-layer plates under axial compression.

The literature is rich in the study of analysis of laminated, composite shells[5] and in the analysis and optimization of stiffened, orthotropic shells.[6] Little information is available, however, on the optimal design of laminated composite shells with the exception of Khot et al.,[7] who consider structures with three layers of variable thickness having ply angles of 0, 45, or 90 deg. Thus the optimization of laminated composite shells is still largely unexplored. The primary purposes of this paper are to examine methods for the determination of optimal or near optimal ply angle variation across the thickness of such shells, and to determine the nature of the improvement in performance that may be expected from optimal configurations.

The methods and conclusions, with respect to out-of-plane loads, of the plate studies of Ref. 4 are largely applicable to shells as well. However, although the buckling, frequency, and stress behavior of flat plates under in-plane loading cannot be improved by variable ply angle construction, shell curvature allows exploitation of such variation. This paper examines for the first time the optimization of the ply angle distribution across the thickness of composite shells. For simplicity this early study treats cases involving the buckling and free vibration behavior of simply supported, thin cylindrical shells of symmetric, orthotropic, angle-ply, filamentary, composite construction where each layer has the same matrix and filament spacing and material.

II. Problem Formulation

Consider an angle-ply shell as shown in Fig. 1 fabricated of a filamentary laminated composite material where ply orientation changes only in the thickness direction and each adjacent laminate set has a ply angle $+\phi$ and $-\phi$ measured from a line on the midsurface parallel to the cylinder axis. The problem is to find the optimal function $\phi(z)$. This problem can be approached in several ways. Two schemes will be considered here both involving sequences of functions. The first employs a series (S^1) of piecewise constant terms to approximate $\phi(z)$. The other employs a continuous, piecewise linear series of terms (S^2). These approximations are illustrated in Fig. 2 for the case of a variation symmetric about the midplane.

The solution to finding the optimal $\phi(z)$ function can now be approached by considering a sequence of problems where for each problem of the sequence having k function segments there are $I(k)$ variables. Starting from some arbitrary number of segments (usually $k=1$) a mathematical programming (MP) problem in I variables is formulated and solved. The number of segments is then increased and the process repeated until the improvement produced by additional segments is less than a specified value.

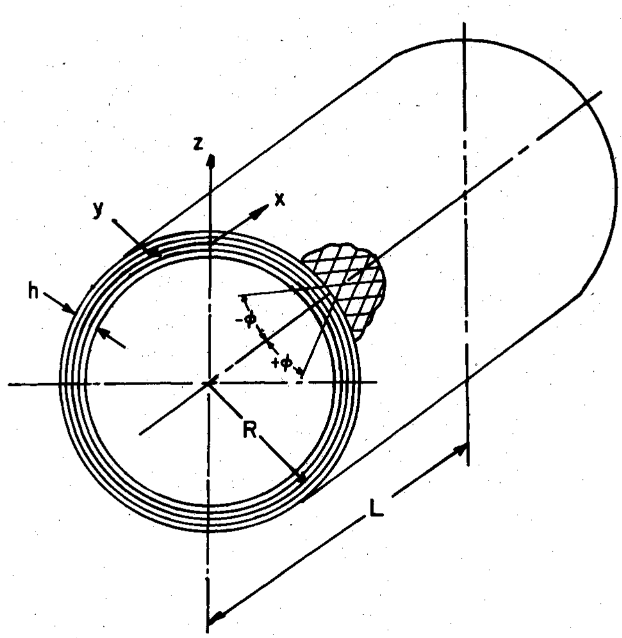

Fig. 1 Shell parameters.

If the filament orientation changes only in the thickness direction, the mechanical properties matrix Q for a given material at a point z is only a function of $\phi(z)$. Where $\phi(z)$ is approximated by a series of segments in segment i, the matrix $Q_i = Q_i[\phi(z)] = Q_i(\phi)$. For constant ply angle segments (S^1),

$$\phi_i(z) = \phi_i \qquad z_{i-1} \leq z \leq z_i \qquad i = 1,2,..,N \tag{1}$$

where N is the number of segments (see Fig. 2a). For the variable ply angle segments (S^2) in a segment i, ϕ may be approximated by

$$\phi_i(z) = \text{sign}\left\langle z - \frac{z_i + z_{i-1}}{2} \right\rangle c_i(z) \tag{2}$$

where $\phi_i(z)$ is the approximating linear function in the segment i (see Fig. 2b). If $N \gg 1$ and $T = \max t_i/h \ll 1$ where $t_i = |z_{i-1} - z_i|$ then because the overall stiffness

$$\bar{Q} = \lim_{T \to 0} Q_N, \quad Q_N = \sum_{i=1}^{N} \frac{1}{t_i} \int_{t_i} Q_i \mathrm{d}z \tag{3}$$

the material properties for a finite number of segments can be approximate from the second of Eqs. (3) by setting $\bar{Q} = Q_N$.

For a construction where $\phi(z)$ is symmetrical with respect to the midsurface of the cylinder,

$$\phi(-z) = \phi(z), \quad -h/2 \leq z \leq h/2 \tag{4}$$

The structural stiffness matrix for this case takes the form

$$\begin{vmatrix} A(\phi) & 0 \\ 0 & D(\phi) \end{vmatrix} \tag{5}$$

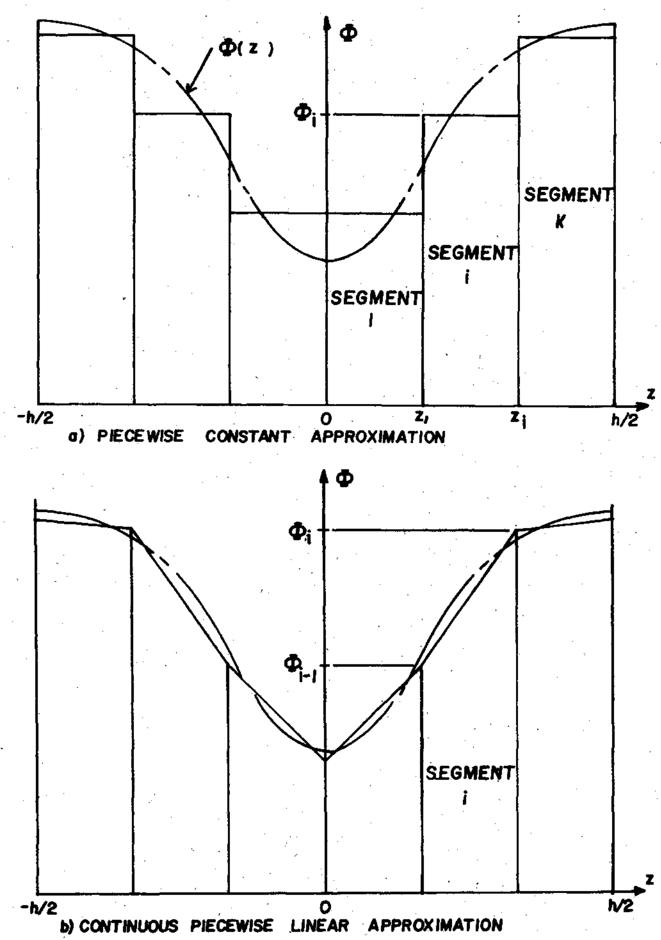

Fig. 2 Approximations to ply angle distribution function $\phi(z)$.

where

$$A(\phi),D(\phi) = \int_{-h/2}^{h/2} Q[\phi(z)][1,z^2]\mathrm{d}z \tag{6}$$

The problem can now be stated as follows: Find the function $\phi^*(z)$ such that

$$\mathcal{L} = \begin{cases} f^* = f[\phi^*(z)] = \min f[\phi(z)] \\ g_j[\phi(z)] \le 0 \qquad j = 1,2,\ldots,J \end{cases} \tag{7}$$

where f is the objective function and g_j the constraints.

If $\phi(z)$ is represented as the limit of a sequence of the series approximations defined by

$$\mathbf{\Phi}_k = \{\phi_k(z), z_k\} \tag{8}$$

where $\mathbf{\Phi}_k$ contains I variables and every $\phi_i(z)$ in $\mathbf{\Phi}_k(z)$ is determined in the interval z_i to z_{i+1}, with $z_0 = 0$ and $z_k = h/2$, then the problem (ϕ) can be replaced by a sequence $\mathcal{L}_k = \mathcal{L}(\mathbf{\Phi}_k)$ and f^* approximated by f_k^*, where

$$f_k^* = f(\mathbf{\Phi}_k^*) \tag{9}$$

Where the approximating sequence S^1 is employed and the $\phi_i(\epsilon)$ are constant, $\mathbf{\Phi}_k$ can be represented as

$$\mathbf{\Phi}_k^1 = \{\phi_1,\phi_2,\phi_3,\ldots,\phi_k,\ t_1,t_2,\ldots,t_{k-1}\} \tag{10}$$

where t_i is the thickness of segment i and is given by

$$t_i = z_{i-1} - z_i \qquad i = 1,2,\ldots,k-1 \tag{11}$$

Here one has $I^1 = 2k-1$ variables and with a symmetrical construction one has $N^1 = 2k-1$ "constant angle-ply segments."

Where approximating sequence S^2 is employed, the $\phi_i(z)$ are piecewise-linear continuous functions, then

$$z_i = \sum_{p=1}^{i} t_p, \qquad z_0 = 0 \tag{12}$$

and

$$\phi_i(z_{i-1}) = \phi_{i-1}(z_i) \qquad i = 1,2,\ldots,k-1 \tag{13}$$

Now if the components of $\mathbf{\Phi}_k$ associated with the ply angles are

$$\mathbf{\Phi}_i = \phi_{i-1}(z_{i-1}),\ \ \mathbf{\Phi}_1 = \phi_0(0),\ \ \mathbf{\Phi}_{k+1} = \phi_k(z_k) \tag{14}$$

then Eq. (8) may be written as

$$\mathbf{\Phi}_k^2 = \{\phi_0,\phi_1,\phi_2,\ldots,\phi_k,\ t_1,t_2,\ldots,t_{k-1}\} \tag{15}$$

and thus one has $I^2 = 2k$ design variables and $N^2 = 2k$ "variable angle-ply segments."

Now solve a sequence of problems $\mathcal{L}_k$, $k = 1,2,\ldots$ until

$$|(f_{k-1}^* - f_k^*)/f_k^*| < \epsilon \tag{16}$$

where ϵ is arbitrarily small. Let this value of k be called k^*.

For shell design the MP problem often takes the form of the generalized mini-max problem[8]; find

$$f_k^* = \min_{\mathbf{\Phi}_k} \{\max_{x} F(\mathbf{\Phi}_k,\mathbf{x})\} \tag{17}$$

subject to

$$g_j = \max_{\bar{x}} \{G(\mathbf{\Phi}_k,\mathbf{x})\} \le 0 \tag{18}$$

The minimization with respect to control or design variables $\mathbf{\Phi}_k$ and maximization with respect to the state or behavior variables $\mathbf{x}$ problems will be referred to here as the design and behavior optimization problems, respectively.

Equations (17) and (18) reduce to the conventional MP form given in Eq. (7) where there is no need to solve the behavior optimization problem. One encounters the behavior optimization problem in structural design in cases where one must determine a critical failure mode or the location of a critical failure point. For example, such problems arise where it is necessary to determine the point of maximum stress, in which case $\mathbf{x}$ are space variables,[4] or where it is necessary to determine the critical buckling mode integers, in which case the behavior variables are n and m. The mini-max objective function form of Eq. (17) is often encountered in the "maximum performance" type problem such as the maximum frequency separation problem treated by Bronowicki et al.,[9] and later by Pappas,[10] or the maximum strength plate problem treated by Hirano[2] and the later studies of Refs. 3 and 4.

In the formulation above the design variables are considered continuous since the purpose of the formulation is to obtain an approximation to the optimal design. The design solution vector $\mathbf{\Phi}_k^*$ must be transformed to a real design by some appropriate method which considers the finite thickness of the laminae. Thus the procedure described here, although useful for design and study of optimal laminated composite shells, will not by itself develop an actual discrete optimal ply angle configuration. Rather the procedure will define an approximation to the optimal ply angle function, which the designer, or additional design algorithm, can then use to specify a near optimal design.

III. Buckling and Vibration Formulations

Two problems are studied here: The selection of the ply angle function $\phi(z)$ to maximize 1) the minimum natural frequency of vibration of circular cylinders made of a linear, elastic, laminated composite material as described above and 2) the buckling resistance of such shells under various loading conditions. For thin, simply supported cylindrical shells this first problem may be posed as follows: For both sequences of series S^r, $r = 1,2$, find

$$\max_{\mathbf{\Phi}_k^r} \{\min_{m,n} [\omega(\mathbf{\Phi}_k^r,m,n)] \quad \text{and} \quad k^* \tag{19}$$

starting with $k = 1$. The solution is constrained by

$$\left(\sum_{i=1}^{k-1} t_i\right) - h/2 < 0 \tag{20}$$

and regional constraints

$$\begin{aligned} & t_i > 0 && i = 1,2,\ldots,k-1 \\ & -90\,\text{deg} < \phi_\ell < 90\,\text{deg} && \ell = 1-1 = 0,1,2,\ldots,k \end{aligned} \tag{21}$$

The behavior variables n and m are integers with

$$n \ge 0, \qquad m \ge 1 \tag{22}$$

It may be seen that Eq. (19) is of the same form as Eq. (17) where $\omega(\mathbf{\Phi}_k^r,m,n) = -F(\mathbf{\Phi}_k,\mathbf{x})$ after noting that the min $(\omega) = \max(-\omega)$.

The second or buckling problem for such shells is of similar form except that Eq. (19) is replaced by

$$\max_{\mathbf{\Phi}_k^r} \{\min_{m,n} [q(\phi_k^r,m,n)]\} \tag{23}$$

where

$$q = \frac{B(\mathbf{\Phi}_k^r,m,n)}{k_1(\pi m/a)^2 + k_2(n/R)^2} \tag{24}$$

Table 1 Optimal variable angle-ply, constant angle-ply, and reference shells (frequency problem)

X	k	$\bar{\omega}_1\ (m,n)$	$\bar{\omega}_2\ (m,n)$	$\bar{\omega}_3\ (m,n)$	ϕ_0, deg	ϕ_1, deg	ϕ_2, deg	ϕ_3, deg	$2t_1/h$	$2t_2/h$
Constant angle ply (S^1)										
60	1	2.17(1,0)	2.17(1,5)	2.20(1,4)	—	46	—	—	—	—
	2	2.58(1,5)	2.66(1,4)	2.84(1,1)	—	20	52	—	0.43	—
	3	2.59(1,5)	2.66(1,4)	2.88(1,6)	—	15	40	53	0.31	0.60
240	1	3.89(1,4)	3.89(1,3)	5.10(1,5)	—	41	—	—	—	—
	2	5.32(1,4)	5.32(1,3)	7.8(1,5)	—	20	65	—	0.54	—
	3	5.35(1,4)	5.35(1,3)	7.07(1,5)	—	13	34	68	0.35	0.24
1500	1	8.8(1,2)	10.2(1,1)	13.1(2,1)	—	90	—	—	—	—
	2	14.56(1,2)	14.56(1,3)	26(2,1)	—	23	90	—	0.82	—
	3	14.57(1,2)	14.57(1,3)	25.6(2,3)	—	23	87	84	0.82	0.15
Variable angle ply (S^2)										
60	1	2.58(1,5)	2.68(1,4)	2.88(1,6)	12	63	—	—	—	—
	2	2.59(1,5)	2.66(1,4)	2.88(1,6)	0	46	57	—	0.51	—
	3	2.59(1,5)	2.66(1,4)	2.88(1,6)	0	20	46	57	0.35	0.30
240	1	5.37(1,4)	5.37(1,3)	7.07(1,5)	0	80	—	—	—	—
	2	5.37(1,4)	5.37(1,3)	7.07(1,5)	0	69	86	—	0.87	—
1500	1	14.15(1,2)	14.15(1,3)	26(2,3)	0	68	—	—	—	—
	2	14.66(1,2)	14.66(1,3)	25.7(2,3)	0	28	87	—	0.63	—
Reference (type A)										
60		2.5(1,5)								
240		5.21(1,4)	—	—	0	90	—	—	—	—
1500		13.6(1,2)								

Table 2 Optimal variable angle-ply, constant angle-ply, and reference shells (buckling problem, lateral loading $k_1 = 0$, $k_2 = 1$)

X	k	$\bar{q}_1\ (m,n)$	$\bar{q}_2\ (m,n)$	$\bar{q}_3\ (m,n)$	ϕ_0, deg	ϕ_1, deg	ϕ_2, deg	ϕ_3, deg	$2t_1/h$	$2t_2/h$
Constant angle ply (S^1)										
60	1	0.195(1,5)	0.195(1,6)	0.217(1,4)	—	48	—	—	—	—
	3	0.250(1,5)	0.250(1,6)	0.291(1,7)	—	20	64	—	0.368	—
	5	0.255(1,5)	0.255(1,6)	0.296(1,7)	—	14	55	68	0.30	0.37
240	1	1.21(1,4)	1.70(1,5)	2.53(1,6)	—	75.3	—	—	—	—
	3	1.99(1,5)	2.33(1,4)	2.70(1,6)	—	26	90	—	0.406	—
	5	1.99(1,5)	2.33(1,4)	2.70(1,6)	—	26	90	90	0.406	0.583
1500	1	19.6(1,2)	36.0(1,3)	39.0(2,3)	—	90	—	—	—	—
	3	39.0(1,3)	39.3(1,2)	65.1(1,4)	—	22	89.7	—	0.33	—
	5	39.0(1,3)	39.0(1,2)	65.2(2,3)	—	22	89.8	89.8	0.39	0.518
Variable angle ply (S^2)										
60	2	0.256(1,5)	0.256(1,6)	0.299(1,7)	9	85	—	—	—	—
	4	0.256(1,6)	0.256(1,6)	0.299(1,7)	7	77	73	—	0.86	—
	6	0.257(1,5)	0.257(1,6)	0.299(1,7)	0	54	66	75	0.50	0.21
240	2	1.91(1,4)	2.28(1,5)	3.0(1,3)	3	90	—	—	—	—
	4	2.00(1,4)	2.41(1,3)	2.68(1,5)	0	90	90	—	0.64	—
	6	2.00(1,4)	2.41(1,3)	2.69(1,5)	0	46	90	90	0.41	0.13
1500	2	33.5(1,3)	33.5(1,2)	56.0(1,4)	22	90	—	—	—	—
	4	38.3(1,3)	38.3(1,2)	64.0(2,3)	0	90	90	—	0.63	—
Reference construction										
60		0.253(1,5)								
240	2	1.89(1,4)	—	—	0	90	—	—	—	—
1500		31.6(1,3)								

and B is a buckling factor. The axial line load N_x is given by $N_x = k_1 q$ and the circumferential line load N_y is given by $N_y = k_2 q$.

These problems are employed to illustrate the application of the optimization problem formulation, test the optimization procedure, and generate useful new data on the nature of optimal variable ply angle shells. The vibration and buckling forms are well adapted to these purposes and, due to the simplicity of the behavior equations used, allow optimal parametric studies at relatively modest cost.

The "maximum performance" form of the structural design problem used here, as in the earlier laminated composite plate study of Hirano,[2] is chosen since it provides direct performance comparison data and other useful information on the characteristics of optimal laminated composite shells. Furthermore, the generation of data on the optimal properties of such shells requires a large number of optimization runs. Thus an efficient problem form is important. This maximum performance form utilizes only linear constraints while the "minimum weight" form more frequently used in structural design contains nonlinear constraints. Since most nonlinear MP procedures, including the one used here, move along the contraints in the optimal search and since linear constraints are easier to follow than nonlinear constraints the maximum performance form is, therefore, expected to be more efficient. Further, as noted by Haftka and Prasad,[11] the commonly encountered maximum performance form of the type used here is the equivalent of the more widely used minimum weight form. Thus both can be considered to provide solutions to the same problem. For these reasons the maximum performance form better suits the purposes of this study.

IV. Optimization Procedure

The compound algorithm described in Ref. 8 is used for the solution of the problem of Sec. III. A numerical search based on a highly modified Zoutendijk direction finding (ZDF) problem[12] starting from an arbitrary design vector is employed to solve the design optimization problem. The ZDF is modified to allow treatment of multiple objective functions resulting from the coalescence of vibration and buckling modes near the optimum.[10] The ZDF problem formulation is further modified to more effectively treat multiple active objective functions resulting from a mini-max objective function.[8]

The behavior problem uses the integer search procedure of Pappas and Amba-Rao[14] to locate the vibration or buckling minima. This procedure seems completely reliable for the vibration problem forms. The occasional extreme flatness of the buckling load surface, however, can result in failure to locate the exact minimum values of n and m for near optimal designs. Thus at search termination an exhaustive search of the buckling surface can be made. If the minimum is not in agreement with that produced by the procedure of Ref. 14 then the entire search can be restarted from the termination point using an exhaustive search for the behavior optimization problem solution. In the examples studied here such restart would always produce only very minor improvement in buckling resistance (less than 0.5%) and does not, as a practical matter, seem needed for design purposes. Thus the integer search of Ref. 14 seems suitable for design purposes on these problem forms unless oscillation of the optimization procedure resulting from a failure to locate the behavior optimum is encountered. If this occurs the run must be terminated or an exhaustive search employed.

V. Results

The above formulation and procedure is applied to the problem of the determination of the optimal ply angle distribution in simply supported cylindrical shells with the following lamina properties: $E_1 = 207$ GPa (30×10^6 psi), $E_2 = 5.17$ GPa (0.75×10^6 psi), $G_{1,2} = 2.69$ GPa (0.375×10^6 psi), and $\nu_{1,2} = 0.25$. The study employs a shell thickness parameter $R/h = 30$ and length parameters $X = L^2/Rh$ of 60, 240, and 1500, representing short, intermediate, and long shells, respectively. Both series approximations S^1 and S^2 are used for the optimal vibration and buckling performance studies. The vibration and buckling analysis is performed using equations given by Ambartsumyan,[15] with the stiffness matrix formulation for constant angle-ply segments from Ref. 2, and for variable angle-ply segments from Refs. 3 and 4. In addition, two variable angle-ply reference configurations, one with a zero ply angle at the midplane and 90 deg at the inner and outer surfaces (type A), and the other with 90 deg at the midplane and 0 deg at the outside and inside surfaces (type B), are also investigated.

The results of the optimal vibration performance studies are given in Table 1, which presents the frequencies and associated mode shapes (m,n) and the approximation to the optimal ply angle configuration for various k. The performance of the type A reference configuration is also presented. Here $\bar{\omega}_j = \omega_j/(\pi^4 h^3 E_1/L^4)$, $j = 1,2,3$ where the ω_j are the natural frequencies of vibration with $\omega_1 < \omega_2 < \omega_3$ and ω_1 the minimum natural frequency.

It may be seen that, as expected, the variable angle-ply segment approximation (S^2) converges more rapidly than the constant angle-ply segment approximation (S^1) in all cases. For a given k the S^2 approximation always yields a shell with a higher $\bar{\omega}_1$. In all instances the S^2 approximation with k different segments is equal or superior to the S^1 approximation using $k+1$ different approximating segments. The better S^2 approximation produces near optimal results with $k=1$. For this approximation, where $\epsilon = 0.01$, search termination occurs with $k^* = 2$ for the case of the short and intermediate shells and $k^* = 3$ for the long shells. Further, it may be seen by examining the frequencies that the optimal variable angle-ply shells (S^2 approximation, $k = 2$ or 3) are significantly superior in performance to optimal conventional angle-ply shells using a single value of ply angle (S^1 approximation, $k=1$). The performance gain increases with shell length, being about 20, 40, and 70% in ω_1 and about 40, 90, and 180% in ω_1^2, for short, intermediate, and long shells, respectively. The type A reference configuration is seen to be comparable in performance to the optimal shells, being particularly effective for short shells.

Contour map plots of several of the $\omega(n,m)$ functions[14] for the near optimal designs show that typically such functions are unimodal with two adjacent, essentially equal frequency modes, one of which is the lowest frequency mode. All other modes are associated with distinctly higher frequencies. Thus the location of the minimum natural frequency (behavior optimization problem solution) is easily and reliably accomplished using the search procedure of Ref. 14. The presence of more than one critical frequency mode demonstrates the need for an optimization procedure capable of treating more than one objective function. Where ω_1 and ω_2 are essentially equal, both must be increased by a search move or the move may fail. Where $\omega_1 \approx \omega_2$, a move increasing only the frequency of the mode associated with the current ω_1 may reduce the frequency of the mode associated with the current ω_2, thereby resulting in a false move that reduces rather than increases the minimum natural frequency.

In the optimal buckling resistance study a number of loading conditions are investigated, including axial ($k_1 = 1$, $k_2 = 0$), equal ($k_1 = k_2 = 1$), hydrostatic ($2k_1 = k_2 = 1$), and lateral ($k_1 = 0$, $k_2 = 1$) loading so as to provide information on the nature of the changes in optimal shell characteristics resulting from differing loading conditions. The equal loading condition is included in the study since it is intermediate between the lateral and axial loading conditions. Hydrostatic loading, of course, provides a much larger lateral than axial loading fraction. The equal loading condition can be encountered in underwater situations where hydrostatic pressure and axial loading are combined, such as in an axial load supporting, evacuated tube in a submerged structure.

These results are presented in Tables 2-4, which present essentially the same type of data as Table 1 (q replacing ω) for lateral, equal, and axial loading, respectively. Hydrostatic loading results are not presented in tabular form since they are similar to those for equal loading, which are seen to be quite similar to the lateral loading results. Thus shells with substantial lateral loading fractions have similar characteristics. Only the axial loading condition produces optimal shell characteristics substantially different than those with significant lateral loading.

Examining Tables 2 and 3 and comparing them to Table 1 shows that the results of the studies of shells with significant lateral load are very similar to the optimal vibration performance study with respect to convergence properties of the approximations used, performance increases obtained by a variable angle-ply construction (using ω^2 as the comparison base since ω^2 rather than ω is analogous to q), and topology of

Table 3 Optimal variable angle-ply, constant angle-ply, and reference shells (buckling problem, equal loading $k_1 = k_2 = 1$)

X	k	$q_1(m,n)$	$q_2(m,n)$	$q_3(m,n)$	ϕ_0, deg	ϕ_1, deg	ϕ_2, deg	ϕ_3, deg	$2t_1/h$	$2t_2/h$
Constant angle ply (S^1)										
60	1	1.42(1,6)	1.42(1,5)	1.69(1,7)	—	48	—	—	—	—
	2	1.86(1,5)	1.86(1,6)	2.21(1,7)	—	20	65	—	0.37	—
	3	1.87(1,5)	1.87(1,6)	2.29(1,7)	—	14	55	69	0.30	0.37
240	1	2.43(1,4)	2.43(1,3)	3.26(1,5)	—	51	—	—	—	—
	2	4.26(1,4)	4.50(1,3)	6.06(1,5)	—	26	90	—	0.41	—
	3	4.26(1,4)	4.50(1,3)	6.06(1,5)	—	26	90	90	0.41	0.59
1500	1	4.93(1,2)	4.94(1,3)	7.83(1,4)	—	45	—	—	—	—
	2	14.56(1,3)	14.58(1,2)	21.4(2,3)	—	22	90	—	0.43	—
	3	14.56(1,3)	14.58(1,2)	21.4(2,3)	—	22	90	90	0.43	0.45
Variable angle ply (S^2)										
60	1	1.88(1,5)	1.88(1,6)	2.18(1,7)	10	72	—	—	—	—
	2	1.89(1,5)	1.89(1,6)	2.15(1,7)	0	56	63	—	0.62	—
	3	1.89(1,5)	1.89(1,6)	2.18(1,7)	0	54	60	60	0.59	0.30
240	1	4.05(1,4)	5.12(1,5)	5.80(1,3)	3	90	—	—	—	—
	2	4.27(1,4)	4.66(1,3)	6.03(1,5)	0	90	90	—	0.64	—
	3	4.27(1,4)	4.66(1,3)	6.03(1,5)	0	90	90	90	0.64	0.35
1500	1	12.58(1,3)	12.59(1,2)	20.7(2,3)	20	90	—	—	—	—
	2	14.26(1,2)	14.27(1,3)	22.0(2,3)	0	90	90	—	0.68	—
	3	14.26(1,2)	14.60(1,3)	21.53(2,3)	11	26	90	90	0.36	0.12
Reference (type A)										
60		1.80								
240	2	4.04	—	—	0	90	—	—	—	—
1500		11.0								

Table 4 Optimal variable angle-ply, constant angle-ply, and reference shells (buckling problem, axial compression $k_2 = 0$, $k_1 = 1$)

X	k	$q_1(m,n)$	$q_2(m,n)$	$q_3(m,n)$	ϕ_0, deg	ϕ_1, deg	ϕ_2, deg	ϕ_3, deg	$2t_1/h$	$2t_2/h$
Constant angle ply (S^1)										
60	1	2.48(1,3)	2.50(1,4)	—	—	22	—	—	—	—
	2	5.31(1,5)	5.33(3,0)	5.39(3,1)	—	89	36	—	0.31	—
	3	5.48(1,5)	5.48(3,1)	5.48(3,2)	—	84	18	47	0.36	0.24
1500	1	53.40(7,0)	53.55(3,4)	53.58(4,5)	—	12	—	—	—	—
	2	109.0(2,3)	109.0(1,3)	109.0(3,4)	—	90	38	—	0.30	—
	3	124.8(15,0)	124.8(16,0)	124.8(3,4)	—	73	0	42	0.30	0.47
Variable angle ply (S^2)										
60	1	5.03(1,5)	5.03(2,0)	5.03(3,0)	68	17	—	—	—	—
	2	5.36(1,5)	5.36(1,5)	5.37(3,1)	0	90	17	—	0.38	—
	3	5.40(3,2)	5.40(1,5)	5.40(3,1)	0	90	90	21	0.35	0.05
1500	1	101.39(1,3)	101.42(3,4)	101.46(3,4)	90	17	—	—	—	—
	2	129.6(6,7)	129.6(6,8)	129.8(6,9)	0	90	90	—	0.62	—
	3	129.9(3,4)	130.0(16,0)	130.2(15,0)	0	90	8	0	0.558	0.447
Reference (type A)										
60		2.84(4,0)	—	—	0	90	—	—	—	—
1500		76.4(21,0)						—		
Reference (type B)										
60		4.18(1,5)	—	—	90	0	—	—	—	—
1500		79.0(4,0)								

the behavior function. Thus the earlier comments on the results of the vibration performance study apply to the buckling studies for the cases where loading includes substantial lateral pressure. Even the optimal ply angle distributions are similar, the primary difference being that the slope of $\phi(z)$ is somewhat larger in the optimal buckling resistant designs (see Fig. 3).

The axial compression studies identify uniquely different optimal shell characteristics. Optimal shells for this loading condition typically have many buckling modes with nearly equal buckling values. A typical contour map for the buckling load $q(n,m)$ surface of a near optimal axially loaded design plotted for all modes associated with $n = 0,1,2,...,10$ and $m = 1,2,...,20$, is shown in Fig. 4. For illustrative purposes, lines of constant buckling load are plotted assuming n and m are continuous rather than integer variables.[14] An examination of the contours shows there are about 20 modes (combination of m and n) with buckling values within 3% of the critical buckling load. The buckling load surface is seen to contain a narrow flat ridge with many nearly equal valued

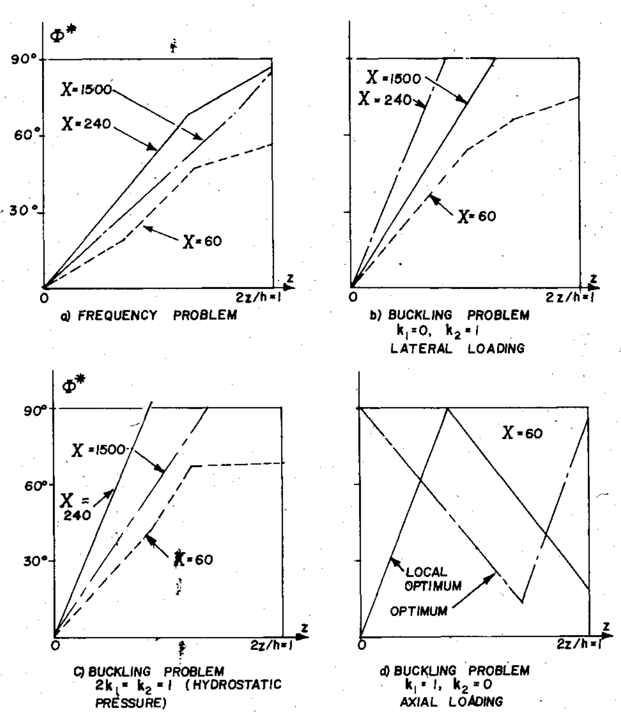

Fig. 3 Typical optimal ply angle distributions.

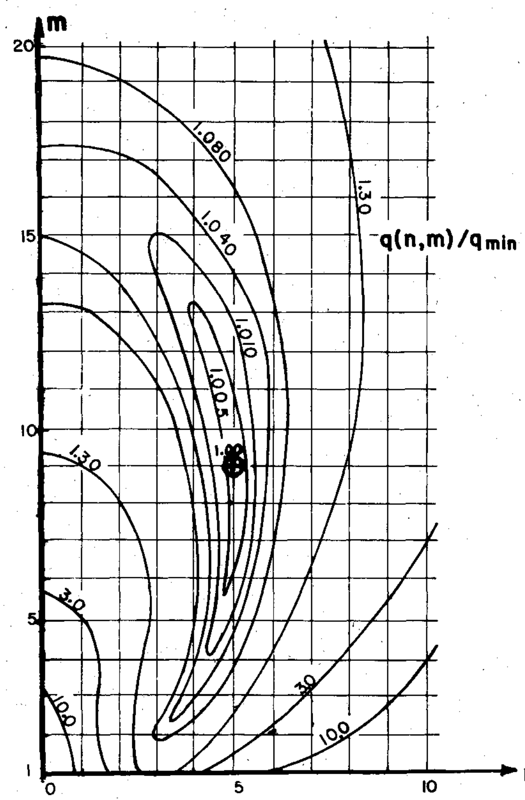

Fig. 4 Contour lines of $q(n,m)$ for near optimal variable angle-ply shell under axial compression.

discrete points, one of which is the optimum. Such a surface is extremely difficult to search reliably using conventional integer search methods. Thus the integer search of Ref. 14 is not reliable on such surfaces. An exhaustive search should be used for the later stages of the optimization process to insure solution of the behavior optimization problem and thus avoid the possible algorithm oscillation which can result from the failure to reliably solve this problem. The presence of so many active buckling modes again demonstrates the critical need for an optimization algorithm capable of treating multiple objective functions if one is to solve the maximum buckling resistance problem.

Alternative starting point runs used to attempt to find alternate local optima indicate none are present in the cases studied except in the axially loaded variable angle-ply segment cases. Here, local optima with critical buckling loads about 2% lower than the global optimum are found to exist. The global optimum for the case of a short shell where the S^2 sequence with $k=2$ is used has $q_1=q_2=q_3=5.48$ with variables $\phi_0,\phi_1,\phi_2,2t_1/h=90$, 6, 90, 0.72 deg. These variable values are seen to be distinctly different than the locally optimal values given in Table 4 for this case. Both configurations are shown in Fig. 3d.

The results for the axially loaded case are different from the other loading conditions in other respects. The performance improvement obtained by use of a variable ply angle is not strongly dependent on shell length, as it is for the other loading conditions, being about 120% for short and 140% for long shells. Further, the nature of the optimal $\phi(z)$ is substantially different. For the axially loaded optimal shells the $\phi(z)$ function typically has a stationary point between the midplane and surface, while in all other cases considered $\phi(z)$ increases monotonically from the midplane to the surface. Thus the characteristics of axially loaded shells are quite different from those with substantial lateral loading.

The type A reference (0, 90 deg) nonoptimal variable angle-ply shells are superior in all buckling studies to the optimal conventional angle-ply shells (S^1 approximation, $k=1$). This reference construction approaches the performance of optimal variable angle-ply shells for short and intermediate shells except for shells under predominately axial load. For axially loaded shells, the type B reference (90, 0 deg) configuration is more effective. The reason for this situation is apparent from Fig. 3, where it may be seen that, except for the axial loading case, the type A reference configuration is a rough approximation to the optimal distribution. For the axially loaded case the type B reference configuration is more like the optimal.

VI. Conclusions

The results of Ref. 4 and this work illustrate the nature of the improved performance over optimal conventional angle-ply construction available by use of an optimal variable angle-ply configuration. The increase in natural frequencies and buckling resistance of shells is substantial in all cases studied and always quite large in intermediate length and long shells. From the stress studies of Ref. 4 one would also expect significantly increased resistance to most stress related failure modes resulting from use of optimal variable angle-ply construction. The continuously variable ply angle approximation formulation is preferable to the discontinuous constant angle-ply segment form due to the superior convergence properties of the former. Experience with the experimental programs used to generate the results indicates that a preliminary design capability based on classical analytical methods of the type used here can produce close approximations to optimal designs at reasonable cost. A typical optimization run using $k=2$ with the S^2 approximating sequence requires about 5 s on an IBM 3701 168 with the experimental computer program designed for the purposes of this study. More carefully prepared software should be considerably more efficient. It should be noted that the rapid convergence of the piecewise linear approximation sequence to the optimal ply angle function allows a close approach to the true optimum with few variables. Thus a near optimum ply angle distribution may be found from the solution of problems with relatively few variables making the use of MP procedures quite economical.

For maximum performance optimization, the frequent presence of multiple active behavior modes demonstrates the need for MP procedures capable of treating multiple objective functions of the form described herein. Furthermore, the need to efficiently and reliably locate critical modes is important in reducing overall computational effort, since due to this mode coalescence it is usually necessary to search for the critical mode at every point of the design variable search to avoid search oscillation. Thus specialized techniques for treatment of such problems are important elements of an efficient design capability. Conventional MP procedures and exhaustive mode searches are either not suitable or poorly suited to these problems.

The solution of the example problems, with the exception of the axially loaded cases, is straightforward, using the procedure given herein. The axially loaded problem is, however, quite formidable due to the presence of multiple local optima and large numbers of active buckling modes. Results indicate however that such problems can be solved routinely by the suggested procedure particularly if an exhaustive search for the critical m and n is used near the end of the design variable search when oscillation due to mode switching is encountered.

An interesting aspect of this study is that quite efficient shells may be obtained by the use of the reference variable angle-ply configurations. Such shells are often near optimal in performance and always superior to optimal conventional angle-ply shells. Furthermore, this performance can be obtained without resort to formal optimization methods in cases where such reference forms are particularly effective. It should be noted that these desirable reference configurations were deduced from the results of the formal optimization studies. That is, the optimization studies suggested desirable design configurations for certain situations. Thus these optimization procedures are not only directly useful for design, in that they generate approximations to optimal configurations, but they are also useful since they increase understanding of the nature of efficient design configurations. This knowledge can then be used for efficient design without direct resort to formal optimization. These formal methods are, however, a key element in the generation of this knowledge.

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References

[1] Housner, J. M. and Stein, M., "Numerical Analysis and Parametric Studies of the Buckling of Composite Orthotropic Compression and Shear Panels," NASA TND-7996, Oct. 1975.

[2] Hirano, Y., "Optimum Design of Laminated Plates Under Axial Compression," *AIAA Journal,* Vol. 17, Sept. 1979, pp. 1017-1018.

[3] Gruni, V. T. and Nshanian, Y. S., "Optimal Selection of Ply Angle Variation," *Yzeemie Vuz ob Mashiostronie,* No. 4, 1980, pp. 21-27.

[4] Nshanian, Y. S. and Pappas, M., "Optimal Design of Laminated Composite Plates," NJIT Report NV-16, Dec. 1981.

[5] Jones, R. M., "Shells," in *Mechanics of Composite Materials,* McGraw-Hill, New York, 1975, p. 311.

[6] Bushnell, D., "Buckling of Shells—Pitfalls for Designers," *AIAA Journal,* Vol. 19, Sept. 1981, pp. 1183-1226.

[7] Khot, N. S., Venkayya, V. B., and Berke, L., "Optimum Design of Composite Structures with Stress and Displacement Constraints," *AIAA Journal,* Vol. 14, Feb. 1976, pp. 131-137.

[8] Pappas, M., "General Procedure for Numerical Design Optimization," NJIT Report NV-18, March 1982.

[9] Bronowicki, A. J., Nelson, R. B., Felton, L. P., and Schmit, L. A., "Optimum Design of Ring-Stiffened Cylindrical Shells," *AIAA Journal,* Vol. 13, Oct. 1975, pp. 1319-1325.

[10] Pappas, M., "Optimal Frequency Separation in Cylindrical Shells," *AIAA Journal,* Vol. 16, Sept. 1978, pp. 999-1001.

[11] Haftka, R. T. and Prasad, B., "Optimum Structural Design with Plate Bending Elements—A Survey," *AIAA Journal,* Vol. 19, April 1981, pp. 517-522.

[12] Zoutendijk, G., *Method of Feasible Directions,* Elsevier, Amsterdam, 1960.

[13] Pappas, M., "Improved Methods for Large Scale Structural Synthesis," *AIAA Journal,* Vol. 19, Sept. 1981, pp. 1227-1233.

[14] Pappas, M. and Amba-Rao, C. L., "A Discrete Search Procedure for the Minimization of Stiffened Shell Stability Equations," *AIAA Journal,* Vol. 8, Nov. 1970, pp. 2093-2094.

[15] Ambartsumyan, S., "Theory of Anisotropic Shells," NASA TTF-118, May 1964.